16. One eigenvalue of $\begin{bmatrix} 1 & 2 & 2 \\ -1 & 4 & 1 \\ 0 & 0 & 3 \end{bmatrix}$ is $\lambda = 3$. A basis for the corresponding eigenspace is

$\mathcal{E}_3 = \mathrm{Null}(\underline{A} - 3I)$

A. $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}$

B. $\left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

C. $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \right\}$

D. $\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}$

E. $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$

Solve $(A - 3I)\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\left[\begin{array}{ccc|c} -2 & 2 & 2 & 0 \\ -1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} \boxed{1} & -1 & -1 & 0 \\ \cancel{-1}\,0 & \cancel{1}\,0 & \cancel{1}\,0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \leftarrow a = b + c$$

b, c free

eigenvectors are $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} b+c \\ b \\ c \end{bmatrix} = \begin{bmatrix} b \\ b \\ 0 \end{bmatrix} + \begin{bmatrix} c \\ 0 \\ c \end{bmatrix}$

$= b\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

Null$(A - 3I)$

$$\mathcal{E}_3 = \underbrace{\left\{ b\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} : b, c \in \mathbb{R} \right\}}_{\mathrm{span}\left\{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right\}}$$

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need 2 lin. indep vectors from here

easy choice: $\left\{ \begin{array}{ll} \underline{v_1} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} & (b=1, c=0) \\ \underline{v_2} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} & (b=0, c=1) \end{array} \right\}$

Another choice could be

$$\left\{ \begin{array}{ll} \underline{v_1} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} & (b=1, c=0) \\ \underline{v_2} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} & (b=1, c=1) \end{array} \right\}$$

9. A tank originally contains 100 gal of water with a salt concentration of $\frac{1}{2}$ lb/gal. A solution containing a salt concentration of 2 lb/gal enters at a rate of 2 gal/min and the well-stirred mixture is pumped out at the rate of 1 gal/min. The amount of salt in the tank after 50 min is

$$c_{out} = \frac{A(t)}{V(t)}$$

A. 0 lb

B. $400 - 350e^{\frac{1}{2}}$ lb

C. e^{-2} lb

D. 100 lb

E. $\boxed{\text{200 lb}}$

$$\frac{dA}{dt} = r_{in}c_{in} - r_{out}c_{out}$$

$$\frac{dA}{dt} = r_{in}c_{in} - r_{out}\frac{A(t)}{V(t)}$$

$$V(t) = V_0 + (r_{in} - r_{out})t$$

$$V_0 = 100$$

$$A(0) = (\text{initial concentration})(\text{initial Vol})$$
$$= \frac{1}{2} \cdot 100 = 50 \text{ lbs}$$

$r_{in} = 2$ $\quad r_{out} = 1$

$c_{in} = 2$ $\quad c_{out} = A(t)/V(t) = \dfrac{A(t)}{100+t}$

$$V(t) = 100 + (2-1)t = \underline{100+t}$$

$$\frac{dA}{dt} = r_{in}c_{in} - r_{out}\frac{A(t)}{V(t)} \qquad y' + Py = Q$$

$$A' = 2 \cdot 2 - \frac{1}{100+t} \cdot A \Rightarrow A' + \frac{1}{100+t}A = 4$$

$P = \dfrac{1}{100+t}$

$Q = 4$

(pick) $\rightarrow$ $\rho = e^{\int P\,dt} = e^{\int \frac{1}{100+t}dt} = e^{\ln(100+t) \cancel{+C}}$ pick $c=0$

$$= (100+t)$$

(all we need is $\rho' = \rho \cdot P$)

$$\rho A' + \rho PA = \rho Q \Rightarrow (\rho \cdot A) = \int \rho \cdot Q\,dt + C$$

$$(100+t)A = \int (100+t) \cdot 4\,dt + C$$

$$(100+t)A = 400t + 2t^2 + C$$

$A(0) = 50$

$$(100+t)A = 400t + 2t^2 + C$$

$$A(0) = 50 \implies (100+0)50 = 0 + 0 + C$$

$$5000 = C$$

$$\boxed{A(t) = \frac{400t + 2t^2 + 5000}{100+t}}$$

$$A(50) = \frac{400 \cdot \cancel{50} + 2 \cdot (50)^{\cancel{2}} + \cancel{(50)}100}{2\cancel{(50)} + \cancel{(50)}1}$$

$$= \frac{400 + 100 + 100}{3} = \frac{600}{3} = 200$$

$$\underline{A(50) = 200}$$

24. The general solution of $\mathbf{x}' = \begin{bmatrix} 3 & 5 \\ -1 & -1 \end{bmatrix} \mathbf{x}$ has the form

A. $c_1 e^t \begin{bmatrix} 2\cos t - \sin t \\ -\cos t \end{bmatrix} + c_2 e^t \begin{bmatrix} \cos t + 2\sin t \\ -\sin t \end{bmatrix}$

B. $c_1 e^t \begin{bmatrix} \cos t \\ 2\cos t - \sin t \end{bmatrix} + c_2 e^t \begin{bmatrix} -\sin t \\ \cos t - 2\sin t \end{bmatrix}$

C. $c_1 e^t \begin{bmatrix} 2\cos t + \sin t \\ -\cos t \end{bmatrix} + c_2 e^t \begin{bmatrix} \cos t + 2\sin t \\ \sin t \end{bmatrix}$

D. $c_1 e^t \begin{bmatrix} \cos t - 2\sin t \\ \sin t \end{bmatrix} + c_2 e^t \begin{bmatrix} 2\cos t + \sin t \\ -\cos t \end{bmatrix}$

E. None of the above.

Find eigenvalues

det $(A - \lambda I)$

$$= \begin{vmatrix} 3-\lambda & 5 \\ -1 & -1-\lambda \end{vmatrix}$$

$$= (3-\lambda)(-1-\lambda) + 5$$

$$= \lambda^2 - 2\lambda - 3 + 5$$

$$= \underline{\lambda^2 - 2\lambda + 2} = 0$$

$$\lambda = \frac{2 \pm \sqrt{4-8}}{2} = \frac{2 \pm \sqrt{-4}}{2} = \frac{2 \pm 2i}{2} = 1 \pm i$$

<u>eigenvalues are $\lambda = 1+i$</u> (mult. 1) $\lambda = 1-i$ (mult. 1)

need eigenvector(s): Solve $\underbrace{\left(\underline{A} - (1+i)I\right)}_{(\underline{A} - \lambda I)} \underbrace{\begin{bmatrix} a \\ b \end{bmatrix}}_{\underline{v}} = \underbrace{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{= \emptyset}$

$$\left[\begin{array}{cc|c} 2-i & 5 & 0 \\ \cancel{-1}\ 1 & \cancel{-2-i}\ +2+i & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 2+i & 0 \\ 2-i & 5 & 0 \end{array}\right]$$

$R_2 \to R_2 - (2-i)R_1$ $\left[\begin{array}{cc|c} \boxed{1} & 2+i & 0 \\ 0 & 0 & 0 \end{array}\right]$ $(2+i)(2-i) = 2^2 - i^2 = 5$

$a + (2+i)b = 0 \implies a = (-2-i)b$

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} (-2-i)b \\ b \end{bmatrix}, \quad b = 1 \quad \begin{bmatrix} -2-i \\ 1 \end{bmatrix} = \underline{v}$$

$$\left(b = -1 \implies \begin{bmatrix} 2+i \\ -1 \end{bmatrix} = v \right)$$

$\lambda = 1+i \qquad \underline{v} = \begin{bmatrix} -2-i \\ 1 \end{bmatrix}$

make $\underline{X}_1 = \operatorname{Re}\left(e^{\lambda t}\underline{v}\right), \quad \underline{X}_2 = \operatorname{Im}\left(e^{\lambda t}\underline{v}\right)$

$$e^{\lambda t}\underline{v} = e^{(1+i)t}\begin{bmatrix} -2-i \\ 1 \end{bmatrix} =$$

$$= e^{t}\left(\cos(t) + i\sin(t)\right)\left(\begin{bmatrix} -2 \\ 1 \end{bmatrix} + i\begin{bmatrix} -1 \\ 0 \end{bmatrix}\right)$$

$$= e^{t}\left((\cos t)\begin{bmatrix} -2 \\ 1 \end{bmatrix} - (\sin t)\begin{bmatrix} -1 \\ 0 \end{bmatrix}\right)$$

$$+ e^{t}\left(i(\sin t)\begin{bmatrix} -2 \\ 1 \end{bmatrix} + i\cos t\begin{bmatrix} -1 \\ 0 \end{bmatrix}\right)$$

$$= \boxed{e^{t}\begin{bmatrix} -2\cos t + \sin t \\ \cos t \end{bmatrix}} + i\boxed{e^{t}\begin{bmatrix} -2\sin t - \cos t \\ \sin t \end{bmatrix}}$$

$\underline{X}_1 =$ Real Part $\qquad \underline{X}_2 =$ Imag. part

$$c_1 e^{t}\begin{bmatrix} -2\cos t + \sin t \\ \cos t \end{bmatrix} + c_2 e^{t}\begin{bmatrix} -2\sin t - \cos t \\ \sin t \end{bmatrix}$$

$\boxed{\text{A. } c_1e^t\begin{bmatrix} 2\cos t - \sin t \\ -\cos t \end{bmatrix} + c_2e^t\begin{bmatrix} \cos t + 2\sin t \\ -\sin t \end{bmatrix}}$

B. $c_1e^t\begin{bmatrix} \cos t \\ 2\cos t - \sin t \end{bmatrix} + c_2e^t\begin{bmatrix} -\sin t \\ \cos t - 2\sin t \end{bmatrix}$

C. $c_1e^t\begin{bmatrix} 2\cos t + \sin t \\ -\cos t \end{bmatrix} + c_2e^t\begin{bmatrix} \cos t + 2\sin t \\ \sin t \end{bmatrix}$

D. $c_1e^t\begin{bmatrix} \cos t - 2\sin t \\ \sin t \end{bmatrix} + c_2e^t\begin{bmatrix} 2\cos t + \sin t \\ -\cos t \end{bmatrix}$

E. None of the above.